EXERCISE – V HINTS & SOLUTIONS

Sol.1 **(a)** $\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0$

$\frac{dy}{dx} = \frac{x \pm \sqrt{x^2 - 4y}}{2}$

or (C) option is statisfying given D.E.

(b) $y^2 = 2c\,(x + \sqrt{c}\,)$

$2y\frac{dy}{dx} = 2c \quad \Rightarrow c = y\frac{dy}{dx}$

$y^2 = 2y\frac{dy}{dx}\left(x + \sqrt{\frac{ydy}{dx}}\right)$

$y^2 = 2yy'x + 2\,(yy')^{3/2}$

$(y^2 - 2yy'x)^2 = 4y^3\,(y')^3$

Order = 1 ; Degree = 3

(c) Equation of Normal at point P (x, y) is

$Y - y = -\frac{dx}{dy}\,(X - x) \qquad ....(1)$

Perpendicular distance = |y|

$$\frac{\left|y + \frac{dx}{dy}.x\right|}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}} = |y|$$

$y^2 + \left(\frac{dx}{dy}\right)^2 x^2 + 2xy\frac{dy}{dx} = y^2\left(1 + \left(\frac{dx}{dy}\right)^2\right)$

$\left(\frac{dx}{dy}\right)^2 (x^2 - y^2) + 2xy\frac{dy}{dx} = 0$

$\frac{dx}{dy}\left[\left(\frac{dx}{dy}\right)(x^2 - y^2) + 2xy\right] = 0$

$\frac{dx}{dy} = 0 \Rightarrow x = c$

$x = 1$

or $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

$y = tx \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$t + x\frac{dt}{dx} = \frac{t^2 - 1}{2t}$

$x\frac{dt}{dx} = \frac{t^2 - 1 - 2t^2}{2t}$

$\frac{2t}{t^2 + 1}\,dt = \frac{-dx}{x}$

$\ell n\,(t^2 + 1) + \ell nx = \ell nC$

$x\frac{(y^2 + x^2)}{x^2} = C$

$(1, 1) \qquad \Rightarrow C = 2$

$y^2 + x^2 = 2x$

Sol.2 $(x^2 + 4y^2 + 4xy)\,dy = (2x + 4y + 1)\,dx$

$\frac{dy}{dx} = \frac{2x + 4y + 1}{(x + 2y)^2}$

$\frac{dy}{dx} = \frac{2\left[x + 2y + \frac{1}{2}\right]}{(x + 2y)^2}$

$x + 2y = t \qquad \Rightarrow 1 + 2\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} - 1 = \frac{4t + 2}{t^2}$

$\frac{dt}{dx} = \frac{t^2 + 4t + 2}{t^2}$

$\frac{dt}{dx} = \frac{(t + 2)^2 - 2}{t^2}$

$\frac{t^2}{(t + 2 + \sqrt{2})(t + 2 - \sqrt{2})}\,dt = dx$

After Integrating

$y = \ell n\,[(x + 2y)^2 + 4(x + 2y) + 2]$

$- \frac{3}{2\sqrt{2}}\,\ell n\left(\frac{x + 2y + 2 - \sqrt{2}}{x + 2y + 2 + \sqrt{2}}\right) + C$

Sol.3 Let x_0 be initial population
y_0 be its initial food
Let avg. consumption be a unit.
Than food Reqd initially = ax_0
$y_0 = ax_0$ (0. 90)
$y_0 = 0.9a\,X_0$ ....(1)
Let x be the population of country in 'n' years.

$\frac{dx}{dn}$ = rate of change of population

$= \frac{3}{100}x = 0.03\,x$

$\frac{dx}{x} = 0.03\,dn$

$\ell nx = 0.03\,dn + C$
$x = Ae^{0.03n}$
At n = 0, $x = x_0 \Rightarrow A = x_0$
$x = x_0\,e^{0.03n}$
'y' be the food production in year t

$y = y_0\left(1+\frac{4}{100}\right)^n = 0.9ax_0\,(1.04)^n$

$y_0 = 0.9ax_0$
Food consumption in the year n is $ax_0\,e^{0.03n}$
Again $y - x \geq 0$
$0.9\,x_0\,a\,(1.04)^n > ax_0\,e^{0.03n}$

$\frac{(1.04)^n}{e^{0.03n}} \geq \frac{1}{0.9} = \frac{10}{9}$

$n[\log(1.04) - 0.03] \geq \log 10 - \log 9$

$n \geq \frac{\log 10 - \log 9}{\log(1.04) - \log 0.03}$

Sol.4 volume of water layer
$dV = \pi(r^2 - h^2)\,dh$
flux $Q(t) = v(t) \times A$

$\Rightarrow (72/10^5)\sqrt{2}\sqrt{g}\sqrt{2-h}\int dt = \pi\int\frac{2^2-h^2}{\sqrt{2-h}}dh$

$\Rightarrow (72/10^5)\sqrt{2}\sqrt{g}\sqrt{2-h}.t + c$

$= -2\pi\left[4/3(2-h)^{3/2} - \frac{(2-h)^{5/2}}{5}\right]$

at $t = 0 : h = 0 \Rightarrow c = \frac{-56\pi\sqrt{2}}{15}$

so $t = \frac{-56\pi\sqrt{2}}{15} \times \frac{10^5}{72\times\sqrt{2}\sqrt{g}} = \frac{7\pi\times 10^5}{135\sqrt{g}}$ sec.

Sol.5 $\frac{dy}{dx} = \frac{x^4 - 1 + 2xy}{1+x^2}$

$\frac{dy}{dx} = (x^2 - 1) + \frac{2x}{1+x^2}\,y$

$\frac{dy}{dx} - \frac{2x}{1+x^2}\,y = x^2 - 1$

I.F. $= e^{-\int\frac{2x}{1+x^2}dx} = e^{-\ell n(1+x^2)} = \frac{1}{1+x^2}$

$\frac{y}{1+x^2} = \int\frac{x^2+1-2}{x^2+1}dx$

$\frac{y}{1+x^2} = \int\frac{x^2+1-2}{x^2+1}dx = \int 1 - \frac{2}{1+x^2}dx$

$\frac{y}{1+x^2} = x - 2\tan^{-1}x + c$

passing through origin $\Rightarrow c = 0$
$y = (x - 2\tan^{-1}x)(1 + x^2)$

Sol.6 $f(x) \geq 0$ and $F(x) \geq 0$ ...(1)
and also $F'(x) = f(x)$ ...(2)
also $f(x) \leq cF(x)$
$\Rightarrow F'(x) \leq cF(x)$ using (2)
$\int_0^\infty \frac{F'(x)}{F(x)} \leq \int_0^x c \qquad \Rightarrow F(x) \leq e^{cx}$
using (1) and (2)
$F(x) = 0$:for all $x \geq 0$
Aliter
$F'(x) \leq cF(x)$
$e^{-cx}\,dy/dx - c.e^{-cx}\,y \leq 0$
$\Rightarrow d/dx\,(e^{-cx}\,y) \leq 0$
Let $h(x) = F(x)\,.\,e^{-cx}\downarrow$
$\Rightarrow h(x) \leq h(0) \quad \Rightarrow h(x) \leq 0 \quad \Rightarrow F(x) \leq 0$

Sol.7 (a) $\frac{dv}{dt} \propto -s$

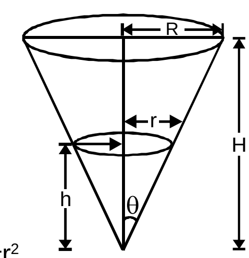
$\frac{dv}{dt} = -ks$

volume of cone $= \frac{1}{3}\pi r^2 h$

surface area $= \pi r^2$

$\tan\theta = \frac{R}{H} = \frac{r}{h}$

$v = \frac{1}{3}\pi r^3\cot\theta$ and $s = \pi r^2$



$\frac{1}{3}\cot\theta\pi 3r^2\frac{dr}{dt} = -k\pi r^2$

$\cot\theta\int_R^0 dr = -k\int_0^T dt$

$R\cot\theta = KT \Rightarrow H = KT$

$\Rightarrow T = H/K$

(b) dy/dx – y > 0 (given)

multiply both sides by e^{-x}

$e^{-x}\, dy/dx - e^{-x}\, y > 0$

$\Rightarrow d/dx\,(e^{-x} y) > 0$

Let $h(x) = P(x)\,.\,e^{-x}\uparrow : h(1) = 0$

$\Rightarrow h(x) > h(1) \quad \Rightarrow h(x) > 0 \quad \Rightarrow P(x) > 0$

Sol.8 **(a)** $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$

$\frac{dy}{1+y} = -\frac{\cos x}{2+\sin x}dx$

$\ell n\,(1+y) = -\log(2+\sin x) + \ell n\, c$

$x = 0,\ y = 1 \Rightarrow c = 4$

$\ell n\,(1+y) = -\ell n\,3 + \ell n\,4$

$1 + y = \frac{4}{3}$

$y = \frac{1}{3}$

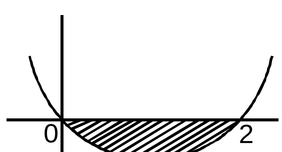
(b) $\frac{dy}{dx} = \frac{(x+1)^2 + y - 3}{(x+1)}$

$\frac{dy}{dx} - \frac{y}{(x+1)} = \frac{(x+1)^2 - 3}{(x+1)}$

$\text{I.F.} = e^{-\int\frac{dx}{x+1}} = \frac{1}{x+1}$

$\frac{y}{(x+1)} = \int\frac{(x+1)^2 - 3}{(x+1)^2}dx$

$= \int 1 - \frac{3}{(x+1)^2}dx$



$\frac{y}{(x+1)} = x + \frac{3}{(x+1)} + c$

(2, 0) passes c = – 3

$y = x(x+1) + 3 - 3(x+1)$

$y = x^2 - 2x$

$\text{Area} = \left|\int_0^2 (x^2 - 2x)\,dx\right| = 4/3$ sq. units

Sol.9 **(a)** $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$

$y = tx \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$t + x\frac{dt}{dx} = \frac{t}{1+t^2}$

$x\frac{dt}{dx} = \frac{t - t - t^3}{1+t^2}$

$\frac{1+t^2}{t^3}dt = -\frac{dx}{x}$

Altar solving and passing through (1, 1)

$\frac{-x^2}{2y^2} + \ell n\, y = -\frac{1}{2}$

$y = e,\ x = x_0$

$-\frac{x_0^2}{2e^2} + 1 = -\frac{1}{2}$

$x_0^2 = 3e^2 \Rightarrow x_0 = \sqrt{3}\,e$

(b) $ydx + y^2dy = xdy$

$ydx - xdy = -y^2dy$

$\frac{ydx - xdy}{y^2} = -dy$

$d\left(\frac{x}{y}\right) = -dy$

$\frac{x}{y} = -y + c \qquad (1, 1) \Rightarrow c = 2$

$y^2 - 2y + x = 0$

$y(-3)$

$y^2 - 2y - 3 = 0 \Rightarrow y = 3, -1$

(c) Length of tangent $= \left|y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\right| = 1$

$\Rightarrow y^2\left(1+\left(\frac{dx}{dy}\right)^2\right) = 1$

$$\frac{dy}{dx} = \pm \frac{y}{\sqrt{1-y^2}}$$

$$\int \frac{\sqrt{1-y^2}}{y}\, dy = \pm x + c$$

$$\sqrt{1-y^2} + \ell n \left| \frac{1-\sqrt{1-y^2}}{y} \right| = \pm x + c$$

(b) $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{y}}$

$$\int \frac{y}{\sqrt{1-y^2}}\, dy = dx$$

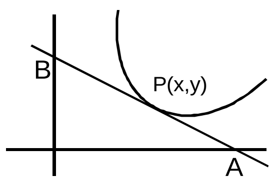
$$-\sqrt{1-y^2} = x + c$$

$(x + c)^2 + y^2 = 1$

centre $(-c, 0)$ radius $= \sqrt{c^2 - c^2 + 1} = 1$

Sol.10 Equation of tangent $Y - y = \frac{dy}{dx}(X - x)$

$\frac{BP}{AP} = \frac{3}{1}$



$B(0, y - mx)$

$A\left(x - \frac{y}{m}, 0\right)$

$\frac{BP}{AP} = \frac{3}{1}$

so, $\frac{dx}{x} = -\frac{dy}{3y}$

$\ell n\, x = -\frac{1}{3} \ell n\, y - \ell n\, c$

$\ell n\, x^3 = -\ell n\, cy$

$\frac{1}{x^3} = cy$ $\quad f(1) = 1 \Rightarrow c = 1$

$y = \frac{1}{x^3}$

Sol.11 (a) $\lim\limits_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$

$x^2 f'(x) - 2x f(x) + 1 = 0$

$\Rightarrow$ $f(x) = cx^2 + \frac{1}{3x}$ $\quad f(1) = 1$

$c = \frac{2}{3}$

$f(x) = \frac{2}{3}x^2 + \frac{1}{3x}$

Sol.12 $x\sqrt{x^2-1}\, dy - y\sqrt{y^2-1}\, dx = 0$

$$\frac{dx}{x\sqrt{x^2-1}} = \frac{dy}{y\sqrt{y^2-1}}$$

$\sec^{-1} x = \sec^{-1} y + c$

$\sec^{-1} 2 = \sec^{-1} \frac{2}{\sqrt{3}} + c$

$c = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

$\sec^{-1} x = \sec^{-1} y + \frac{\pi}{6}$

$y = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$

$\cos^{-1} \frac{1}{x} = \cos^{-1} \frac{1}{y} + \frac{\pi}{6}$

$\cos^{-1} \frac{1}{y} = \cos^{-1} \frac{1}{x} - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\frac{1}{y} = \frac{\sqrt{3}}{2x} - \sqrt{1-\frac{1}{x^2}}\left(\frac{1}{2}\right)$

$\frac{2}{y} = \frac{\sqrt{3}}{x} - \sqrt{1-\frac{1}{x^2}}$

Sol.13 **A,D**

I.F. $= \cos x$

$y \cdot \cos x = \int 2x \sec x \cdot \cos x \cdot dx$

$y \cdot \cos x = x^2 + c,\ c = 0$

$y = x^2 \sec x$